

NUMERICAL CHAOS IN A FRACTIONAL ORDER LOGISTIC MAP

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ABSTRACT. In this paper we investigate a fractional order logistic map and its discrete time dynamics. We show some basic properties of the fractional logistic map and numerically study its period-doubling route to chaos.

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1. INTRODUCTION

The concept of fractional order differentiation and integration is nearly as old as calculus itself [9]. Already in the 17:th century Leibnitz made remarks on the fractional derivative of order $1/2$ [14]. Despite many fundamental results and important definitions found more than 150 years ago by among others Euler and Riemann, the field of applications of fractional calculus has mainly recently started to draw some interest [2, 5, 6, 7, 8, 12]. Applications to physics involve viscoelastic systems and electro-magnetic waves [6], but more recently the dynamics of fractional order dynamical systems, involving fractional mechanics and fractional oscillators has been studied [1, 12]. Fractional differential equations have for example arisen in Quantum Mechanics as well as in fractional generalizations of Einstein field equations [5, 11]. Also fractional generalizations of various famous dynamical systems such as the fractional Lorenz-system, the Rössler systems [6] and Chen's system [7] have been investigated. In this paper we shall investigate the dynamics of a fractional generalization of the classic logistic map [16].

2. DYNAMICS

The discrete dynamics of unimodal maps is well known and has been extensively studied [16]. In particular population dynamics often consider maps like the Ricker-family [16]:

$$(1) \quad R_{\lambda,\beta}(x) = \lambda x e^{-\beta x}, \quad \lambda > 1 \quad \beta > 0,$$

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or the Hassel-family (See [16]):

$$(2) \quad H_{\lambda,\beta}(x) = \frac{\lambda x}{(1+x)^\beta}, \quad \lambda > 1, \beta > 0.$$

However the perhaps most famous unimodal map is the logistic map:

$$(3) \quad Q_\lambda(x) = \lambda x(1-x), \quad \lambda > 0.$$

The logistic map is defined on the half-line $x \in [0, \infty[$, but all of its interesting dynamics takes place on the bounded interval $I = [0, 1]$ for $0 < \lambda \leq 4$. If we consider discrete time evolution by the mapping $Q_\lambda : I \rightarrow I$, letting the state be x_n at time n then we have $x_{n+1} = Q_\lambda(x_n)$ at time $n+1$. In the prescribed interval the parameter λ is increased which results in a period-doubling route to chaos. For a thorough background on unimodal maps and chaos see [16].

3. FRACTIONAL CALCULUS

There are many models of fractional diff-integration, however the more popular ones are the Riemann-Liouville-type operator and Caputo-type operator [10], whereas the latter one is favored in cases of lack of initial conditions. We shall use the Riemann-Liouville fractional operator in order to construct the fractional logistic map. The Riemann-Liouville fractional integral is defined as follows.

Definition 1. If $f(x) \in C([a, b])$ and $a < x < b$ then

$$(4) \quad I_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

where $\alpha \in]-\infty, \infty[$, is called the Riemann-Liouville fractional integral of order α . In the same fashion for $\alpha \in]0, \infty[$ we let

$$(5) \quad D_{a+}^\alpha f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt,$$

which satisfies

$$(6) \quad D_{a+}^0 f(x) = I_{a+}^0 f(x) = f(x),$$

and is called the Riemann-Liouville fractional derivative of order α .

The fractional derivative is a special-case of the fractional integral. A fractional derivative of order $1/2$ is called a semi-derivative and a fractional integral of the same order is called semi-integral [10]. Both the fractional derivative and fractional integral satisfies the important semi-group property:

Theorem 2. For any $f \in C([a, b])$ the Riemann-Liouville fractional integral satisfies

$$(7) \quad I_{a+}^\alpha I_{a+}^\beta f(x) = I_{a+}^{\alpha+\beta} f(x)$$

for $\alpha > 0, \beta > 0$.

A proof may be found in [10]. For more information regarding fractional calculus and its applications see [9, 10, 14].

4. FRACTIONAL DYNAMICS

4.1. Basic properties of the fractional logistic map. Many investigations of fractional dynamical systems contain fractional derivatives in time which is due to their continuous time derivative (See [1, 6, 7]). There exists, however, fractional differential equations in spatial sense, take the fractional Schrödinger equation for example [5]. In our fractional generalization of the logistic map we shall use a spatial fractional derivative. If we let the Riemann-Liouville fractional integral (5) operate on the logistic map (3) we get:

$$(8) \quad I_{0+}^{\alpha} Q_{\lambda}(x) = \frac{\lambda}{\Gamma(\alpha+2)} \left(1 - \frac{2x}{\alpha+2}\right) x^{1+\alpha},$$

which is valid for $\alpha \in]0, \infty[$ (for more details on fractional differentiation see [9]), thus

Definition 3. For all $x \geq 0$, $\lambda > 0$ and $\alpha \in]0, \infty[$:

$$(9) \quad Q_{\lambda}^{\alpha}(x) := \frac{\lambda}{\Gamma(\alpha+2)} \left(1 - \frac{2x}{\alpha+2}\right) x^{1+\alpha},$$

will be called the fractional logistic map (FLM) of order α .

It is left as an exercise to show that the ordinary logistic map (3) follows as a special-case when $\alpha = 0$. The Riemann-Liouville fractional operators have special properties for the parameter-value $\alpha = 1/2$ and are in these cases called semi-integral and semi-derivative respectively [9, 10, 14]. We shall therefore give the definition:

Definition 4. For $x \geq 0$ and $\lambda > 0$ the $\alpha = 1/2$ order fractional logistic map appears like:

$$(10) \quad Q_{\lambda}^{1/2}(x) = \frac{4\lambda}{3\sqrt{\pi}} \left(1 - \frac{4x}{5}\right) x^{3/2},$$

and will be called the semi-logistic map.

Due to the special property of fractional diff-integrals we can obtain the following useful theorem:

Theorem 5. For all $\alpha, n \in]0, \infty[$:

$$(11) \quad \frac{d^n}{dx^n} Q_{\lambda}^{\alpha}(x) = Q_{\lambda}^{\alpha-n}(x).$$

Proof. It follows directly from definition 3 and the semi-group property (Theorem 2). \square

Despite the fact that the fractional logistic map is valid for $\alpha \in [0, \infty]$ it is worth mentioning that probably the most interesting dynamics takes place for $\alpha \in [0, 1]$ since theorem 5 allows for translation to any other domain by N -times integration/differentiation for any $N \in \mathbb{Z}$. We can also give the following theorem:

Theorem 6. For $\alpha \in [0, 1]$:

$$(12) \quad Q_\lambda^\alpha(0) = Q_\lambda^\alpha(1 + \frac{1}{2}\alpha) = 0,$$

holds for all $\lambda \geq 0$.

Proof. $Q_\lambda^\alpha(0) = 0$ follows directly from definition 3. Also we find that $Q_\lambda^\alpha(1 + \frac{1}{2}\alpha) = 0$ by inserting $x = 1 + \frac{1}{2}\alpha$:

$$(13) \quad Q_\lambda^\alpha(1 + \frac{1}{2}\alpha) = \frac{\lambda}{\Gamma(\alpha + 2)} \left(1 - \frac{2 + \alpha}{\alpha + 2}\right) \left(1 + \frac{1}{2}\alpha\right)^{1+\alpha} = 0.$$

Note that $\lambda \geq 0$ is a part of the definition of the fractional logistic map (Definition 3). \square

In order to find fixed points of the fractional logistic map we need the following criterion to be fulfilled:

$$(14) \quad Q_\lambda^\alpha(x) = x,$$

which is an equation on the form:

$$(15) \quad x^{1+\alpha} - \frac{\alpha + 2}{2} x^\alpha + \frac{\Gamma(\alpha + 3)}{2\lambda} = 0.$$

This equation has non-fractional solutions for $\alpha \in [1, 2, 3, \dots]$ which corresponds to simple integer integrations of the logistic map. For the fractional values a general solution is left as an open problem. Below we numerically study the dynamics of the fractional logistic map.

4.2. Period doubling and chaos. With the aid of *Mathematica 5* and software developed in [4] we were able to numerically study the fractional logistic map. As an example we plot the fractional logistic map in the domain $x \in [0, 1], \alpha \in [0, 1]$:

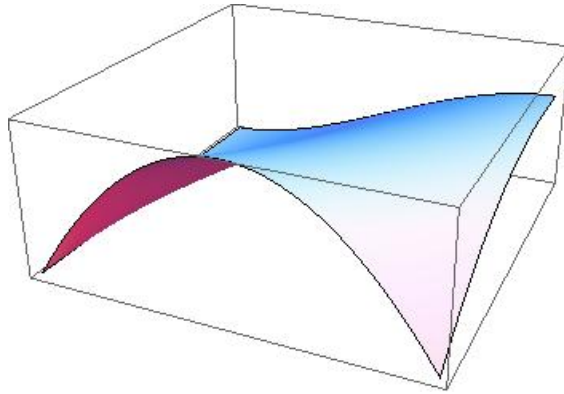
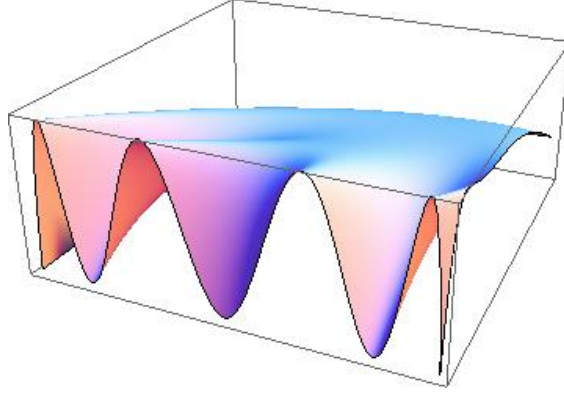
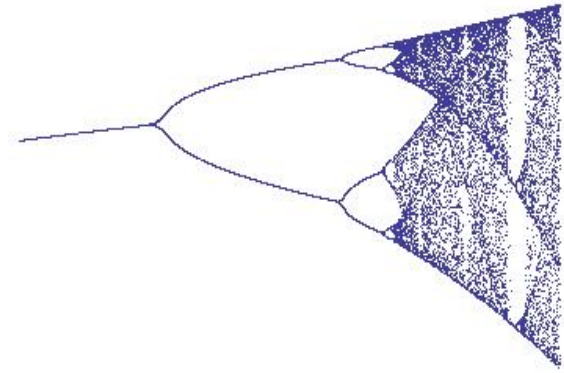


Fig.1. $\alpha \in [0, 1], x \in [0, 1]$.

Note that $x = 1, \alpha = 1$ is a local maximum in the x -direction since the fractional logistic map for $x = 1$ and $\alpha = 0$ is zero due to the property of fractional diff-integrals in theorem 5. As an example we perform three iterates and end up with the following plot:

Fig.2. $\alpha \in [0, 1], x \in [0, 1]$.

In figure 2 we see $Q_\lambda^\alpha(Q_\lambda^\alpha(Q_\lambda^\alpha(x)))$. It is also worth mentioning that the general appearance for any choice of $\alpha \in [0, 1[$ appears similar to the ordinary logistic map (which is the special case when $\alpha = 0$). Indeed, if we further examine the special case $\alpha = 1/2$, the semi-logistic map, then we see that numerically a period-doubling route to chaos appears as λ is increased from 4.5 to 6.1:

Fig.3. $\alpha = 1/2, x \in [0, 3]$.

The appearance of this bifurcation diagram is similar to that observed for the logistic map [16]. Thus we have hinted that the fractional logistic map exhibits a period-doubling route to chaos.

5. DISCUSSION AND OPEN PROBLEMS

In this paper we studied the discrete time evolution of a fractional generalization of the logistic map; the fractional logistic map. We showed some basic properties of the fractional logistic map and numerically hinted a

period-doubling route to chaos for the special case $\alpha = 1/2$. A rigorous proof of this is left for further investigation. The fractional parameter α in the fractional logistic map brings on a whole new range of possibilities for traditional discrete dynamics. The possibility of showing that "fractional chaos" arises when letting the traditional bifurcation parameter be constant while varying the fractional parameter is left as an interesting open problem. Furthermore, the dynamics of the fractional logistic map in the complex plane is also left as an open issue for further investigation. The connection between fractional discrete time dynamics, like the fractional logistic map, and other fractional dynamical systems such as the fractional Rössler and fractional Lorenz systems are also open issues.

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